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On valuation before and after tax in no arbitrage models

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On valuation before and after tax in no arbitrage models:

Tax neutrality in the discrete time model

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Abstract: We establish necessary and sufficient conditions for a linear taxation system to be neutral − within the multi-period discrete time ≫no arbitrage≪ model − in the sense that valuation is invariant to the exact sequence of tax rates, realization dates as well as immune to timing options attempting to twist the time profile of taxable income through wash sale transactions.

Keywords: Tax neutrality, mark-to-market valuation, generalized linear taxation schemes, wash sales.

"In the study of investments, taxes are largely a source of embarrassment to financial economists."

(Introduction to Dybvig and Ross (1986))

"Accordingly, my approach in this chapter is to examine the restrictions on the income measurement rules applicable to financial instruments implied by the requirement that the rules be linear. .... Linearity is a desideratum of a tidy tax system."

(Bradford (2000), p. 373-374)

1 Introduction

Tax considerations play a very important role in most real world investment decisions, whether financial or real. Tax issues, for example the use of tax shields, are also a cornerstone in corporate finance and capital structure theory. It is a remarkable fact, however, that the asset pricing literature by and large ignores taxation issues and has little to say about the effects of different tax systems on asset demand functions and equilibrium market prices. Major textbooks on asset pricing theory and fixed-income analysis do not even have the word ≫taxation≪ or related topics in their index.

There may be a variety of reasons for why the asset pricing literature almost ignores taxation. First, taxation issues are complicated and tend to destroy analytically attractive structures of asset pricing models. Taxation induces a certain individual element into return distributions and pricing relations, because return distributions and discount factors depend on the tax rules. If arbitrage opportunities after tax exist, some constraints like ≫limits to tax deductability≪, ≫limits to short positions≪ or other asset allocation restrictions are necessary in order to limit the extent to which such arbitrage opportunities can be exploited. The resulting market situation will be one marked by corner solutions and clientele effects which are inherently complicated to model.

Second, the tax system may be neutral in the sense that different agents with different tax rates − and maybe even different tax codes − coexist without experiencing arbitrage opportunities. If this is the case one can safely ignore tax considerations. Taxation rules which lead to agreement about asset prices across investors subject to different tax rates and also eliminate the profitability of portfolio dispositions solely made in order to avoid or defer tax payments are known as *neutral taxation systems*.

The purpose of this paper is to examine the characteristics of such neutral taxation systems within the ≫no arbitrage≪ paradigm of mathematical finance. The analysis is carried out for linear taxation systems in order to stay within this paradigm and its characterization of linear pricing operators in terms of equivalent martingale measures.[1)] It obtains a complete characterization of these systems and, hence, is able to identify when taxation issues can be ignored in asset pricing models.

We ask and answer the following simple question: If a set of asset prices and return distributions presents a usual ≫no arbitrage≪ equilibrium before tax, under what conditions will it also be a ≫no arbitrage≪ equilibrium after tax for another investor subject to a linear and symmetric tax schedule. Or, more generally, if a set of asset prices and return distributions presents a ≫no arbitrage≪ equilibrium after tax for *some* taxable investor subject to a linear and symmetric tax schedule, under what conditions will it also be a ≫no arbitrage≪ equilibrium after tax for any other taxable investor subject to a linear and symmetric tax schedule? This is our first criterion for a taxation system to be called neutral and we denote it as ≫valuation neutrality≪. The second criterion is that wash sale opportunities and timing options in a multi-period framework are eliminated. We denote this as ≫holding period neutrality≪.

Clearly, the whole structure of prices and return distributions could − in a more fundamental sense − change as a result of introducing or changing taxes. Obvious channels for such conse-

[1)] In a utility optimization model and even more in a complete equilibrium model arbitrage opportunities after tax can be bounded by having a progressive tax schedule. However, such models require the modeling of the individual's portfolio composition as a whole, cf. e.g. Ross (1987), Dammon and Green (1987) and − recently − Basak and Croituro (2001). This is not the issue here. We restrict ourselves to an investigation of how far the ≫no arbitrage≪ paradigm and the consequent linearity of the pricing operator can be taken.

quences are wealth effects or wealth redistribution effects; additionally, taxation involves a risk sharing mechanism between the government and the taxpaying investor that may be affected when taxes are changed. Such issues are not at stake here, however. Market prices are market prices for taxable investors as well as for tax free investors, and the point of departure is that market prices as well as certain types of investors with different tax rates exist. We want to examine – in a multi-period framework – whether such market prices can be consistent with heterogeneity across investors with respect to taxation in the sense that no arbitrage opportunities exist. In particular, it is comforting to know when the usual ≫before tax≪ asset pricing relations can be valid in an economic setting where none of the investors are actually tax free investors.

Dealing with taxation in a multi-period framework raises the question about how to account for accrued capital income as taxable income. Most of the finance literature deals with a one-period framework, where investment takes place at the beginning of the period and capital income is earned and taxed at the end of the period. The seminal contributions by Schaefer, cf. Schaefer (1981, 1982a,b), belong to this class of models. Other examples are Dybvig and Ross (1986), Dammon (1987), Dammon and Green (1987), Ross (1987), Dermody and Prisman (1988) and Dermody and Rockafeller (1991). One-period models do not give rise to any problems with accruals. Capital income is taxed upon realization and accrued capital income does not exist in a one-period model. The conclusions from the one-period framework, however, cannot be extrapolated to a multi-period framework without careful considerations.

A few papers deal with multi-period problems, but then capital income is taxed whenever it is realized – i.e. at the point in time where it has cash flow consequences. Seminal contributions in this direction encompass Constantinides (1983) and Constantinides and Ingersoll (1984a,b). Other examples are Dermody and Rockafeller (1995), Dammon and Spatt (1996), and Cadenillas and Pliska (1999). For such multi-period models it is well-known that the tax rules give rise to lock-in effects and that tax arbitrage problems are difficult to avoid unless some restrictions – short selling constraints, e.g. – are imposed[2]. Even so, taxable investors may be left with timing options. I.e., for any given series of market price movements a taxable investor can influence to her own advantage the timing of gains and losses as taxable income. A *wash sale* where assets are sold and immediately repurchased with the sole purpose of generating a tax deferral is a well-known example of such timing options.

Neutral taxation system have mostly been studied in the field of public economics. The neutrality property is considered as a normative benchmark with which other taxation systems may be compared and the severeness of deviations measured. However, the public economics literature has only vaguely made use of the analytical techniques developed in finance in order to characterize ≫absence of arbitrage≪ in financial markets. This paper shows that these techniques can be quite powerful outside a narrowly defined territory of finance theory.

The first taxation system to be examined is the mark-to-market value principle. Although the mark-to-market value principle is a special case of the general characterization of neutral linear taxation systems provided in theorem 2 this is done for expositional purposes. The practical problems with implementing such a taxation system – due to the necessity of repeatedly assessing illiquid capital assets – are widely recognized; such assessments are easy to make whenever a transaction realizing a capital gain or loss takes place, but difficult to make for capital assets traded within illiquid markets and without directly observable market prices most of the time. These difficulties may explain why pervasive mark-to-market valuation is seldom

[2] Cf. also footnote 1.

part of real-world tax systems. Despite the practical problems related to an implementation of the mark-to-market valuation principle it appears to be the benchmark type of neutral taxation system in the public economics literature. However, it remains to be demonstrated rigorously within the multi-period ≫no arbitrage≪ paradigm that this taxation principle is neutral. This paper does so.

Following this we derive necessary and sufficient conditions for a linear and symmetric taxation system to be neutral in a multi-period framework. We provide an exhaustive characterization of the possible forms of linear, neutral taxation systems. Interestingly enough, these conditions do not depend on any assumption about completeness of the market, although the necessity part rests on an innocent regularity assumption. The mark-to-market taxation system, the imputed wealth tax system as well as the pure cash flow taxation system due to Brown (1948) all fall into this category as specific examples. The taxation systems proposed by Auerbach (1991), Bradford (1995) and Auerbach and Bradford (2001) do so as well. Simultaneously, we are able to specify exactly the degree of uncertainty allowed for in the interest rate process as well as in the process describing the development of the tax rates over time − an issue largely neglected in the public economics literature.[3)]

The paper is organized as follows. The basic results from the minimal possible ≫no arbitrage≪ model, the one-period binomial model, are stated for future reference in section 2 as an introduction to the more general discussion in the rest of the paper. The main result for the mark-to-market valuation principle in the discrete-time model is stated as theorem 1 in section 3. The possible interpretation of this principle in terms of a realization based tax base is given in section 4. The main results of the paper, necessary and sufficient conditions for neutrality, are stated as theorem 2 in section 5. Finally, section 6 shows an easy way to deal with partial realizations and capital injections as an alternative to tracing each transaction individually. Summary and conclusions are found in section 7.

Proofs are found in Appendix A. In Appendix B we relate the approach taken here to the notation in Auerbach and Bradford (2001). Appendix C completes some details of the derivations in section 6.

2 The one-period binomial model

Consider the usual setup for the binomial model with one risky asset and a riskless asset with interest rate r. In this setup the fundamental pricing relation is stated in terms of the equivalent martingale measure $(q, 1-q)$ as

$$(1 + r)S_0 = qS_u + (1 - q)S_d \tag{1}$$

Let the tax rate be denoted by T; this tax rate is assumed to be the same for all assets under consideration and is applied linearly and symmetrically to a tax base comprised of net earnings. I.e. losses are fully and unconditionally deductible with full tax consequences under all

[3)]In Auerbach (1991), e.g., it is mentioned on p. 171 as a parenthetical remark that "Nothing in the proof depends on either i or t being constant, so variations over time in rates of interest and marginal taxation presents no difficulty". Whether this means time varying or truly stochastic interest rates and/or tax rates is not spelled out in the paper.

circumstances. Then equation (1) can be manipulated into

$$\begin{aligned}(1-T)(1+r)S_0 &= (1-T)\left[qS_u+(1-q)S_d\right] && \Leftrightarrow \\ (1+r(1-T))S_0 &= q(1-T)S_u+(1-q)(1-T)S_d+TS_0 \\ &= q\left[S_u-T(S_u-S_0)\right]+(1-q)\left[S_d-T(S_d-S_0)\right] && (2) \\ &= q\left[(1-T)S_u+TS_0\right]+(1-q)\left[(1-T)S_d+TS_0\right] && (3)\end{aligned}$$

Equation (2) reflects the taxation on a net income basis, which is identical to taxation in accordance with the mark-to-market valuation principle in this simple one-period setup. The equivalent representation in (3) shows that the payment after tax – hence also the tax payment itself – is a linear function of the portfolio values (S_0, S_1); a property to be explored more generally in the following.

The terms in brackets are the payments after tax to the investor under such a linear tax rule and they are denoted here by the superscript $()^{a.t.}$. Hence

$$\begin{aligned}(1+r(1-T))S_0 &= qS_u^{a.t.}+(1-q)S_d^{a.t.} && \Leftrightarrow \\ S_0 &= \left[1+r(1-T)\right]^{-1}\left[qS_u^{a.t.}+(1-q)S_d^{a.t.}\right] \\ &\equiv \left[1+r^{a.t.}\right]^{-1}\left[qS_u^{a.t.}+(1-q)S_d^{a.t.}\right] && (4)\end{aligned}$$

The pricing relation (4) is also valid for the riskless asset with $S_u = S_d = 1+r$ and $S_u^{a.t.} = S_d^{a.t.} = 1+r^{a.t.}$. The market price of the riskless asset is by definition $S_0 = 1$.

Thus, (4) is valid on an after tax basis if and only if (1) is valid on a before tax basis. The Arrow-Debreu prices after tax are equally affected by the factor $(1+r)/(1+r^{a.t.})$ and by the change from before tax prices, $[q/(1+r), (1-q)/(1+r)]$, to after tax prices, $[q/(1+r^{a.t.}), (1-q)/(1+r^{a.t.})]$, *relative state prices* are left unaffected. Simultaneously, the payments after tax decrease. These effects – the effect on discounting, the effect on the Arrow-Debreu prices and the effect on the payments themselves – are precisely counteracting each other, rendering the tax rate irrelevant for asset pricing under a linear and symmetric tax schedule[4]. Observe, however, that taxation does **not** affect the equivalent martingale measure $(q, 1-q)$.

The above derivations produced the »no-arbitrage after tax relation« (4) from the usual »no arbitrage before tax relation« (1). However, there is no need for a before tax pricing relation. If one agent with tax rate T prices assets as shown in (4), then, cf. lemma 1 below, any other agent with a different tax rate $\widehat{T}$ also prices assets in accordance with (4).

Example 1

Let $S_0 = 100$, $S_u = 110$ and $S_d = 95$. With an interest rate of 5% the martingale probability is $q = 2/3$.

The corresponding after tax payments with a tax rate $T = 40\%$ are $S_u^{a.t.} = 106$, $S_d^{a.t.} = 97$ and $r^{a.t.} = 3\%$. Then

$$100 = (1.03)^{-1}\left[\frac{2}{3}106+\frac{1}{3}97\right] \tag{5}$$

[4]This result is also found in Cox and Rubinstein (1985), p. 271-74, as a special case. However, no multi-period analysis after tax is found in Cox and Rubinstein (1985).

In general

$$100 = (1+0.05(1-T))^{-1}\left[\frac{2}{3}(110-10T)+\frac{1}{3}(95+5T)\right] = (1.05-0.05T)^{-1}\left[105-5T\right] \quad (6)$$

∎

The taxation of capital gains and losses as shown in (4) induces one particular type of risk sharing among the investor and the tax authorities which is compatible with valuation neutrality and the absence of arbitrage on an after tax basis. However, other linear taxation schemes and induced risk sharing schemes with this property are possible. One extreme case would be that the investor is allowed to earn exactly the riskless rate of return after tax in both states. In the above example 1 the after tax result would be 103 in both states and all uncertainty about the return on the risky investment will be borne by the tax authorities. Hence, the name ≫government takes all risk≪ is appropriate for this taxation system which is obviously neutral.

Another example would be to tax the investor the amount rTS_0, irrespective of the actual outcome of the risky investment. This would leave the investor with the same risk after tax as before tax in the sense that the spread between the two outcomes as well as the spreads relative to the possible riskless rate of return are the same on an after tax basis as on a before tax basis. This is known as an imputed wealth tax and is also a neutral taxation principle. A third example would be to tax the investor in gross terms by the tax rate $1-(1.025/1.05)\simeq 0.0238$. This would leave more risk with the investor than the mark-to-market valuation principle; the after tax value of the investment is 107.3810 in the up state and 92.7381 in the down state. This is neutral because

$$\begin{aligned}
(1+r)S_0 &= qS^u + (1-q)S^d & \Leftrightarrow \\
\left(1+r^{a.t.}\right)S_0 &= q\frac{1+r^{a.t.}}{1+r}S^u + (1-q)\frac{1+r^{a.t.}}{1+r}S^d & \quad (7)
\end{aligned}$$

independent of T.

All four schemes have the property that the tax to be paid is a linear function of the values (S_0, S_1), and when applied to the riskless asset the return in both states is $r^{a.t.} \equiv r(1-T)$. They also share the property that the investor earns an expected return after tax under the equivalent martingale measure equal to the after tax rate of interest $r^{a.t.}$.

Multiplying (7) by $1-\alpha$ and rearranging terms leads to

$$\left(1+r^{a.t.}\right)S_0 = q\frac{1+r^{a.t.}}{1+r}(1-\alpha)S^u + (1-q)\frac{1+r^{a.t.}}{1+r}(1-\alpha)S^d + \alpha\left(1+r^{a.t.}\right)S_0 =$$

$$q\left[\frac{1+r^{a.t.}}{1+r}(1-\alpha)S^u + \alpha(1+r^{a.t.})S_0\right] + (1-q)\left[\frac{1+r^{a.t.}}{1+r}(1-\alpha)S^d + \alpha(1+r^{a.t.})S_0\right] \quad (8)$$

This equation is trivially true also for $\alpha=1$, which corresponds to the taxation principle ≫government takes all risk≪, where the investor receives $(1+r^{a.t.})S_0$ in both states. $\alpha=T/(1+r^{a.t.})$ corresponds to the mark-to-market valuation taxation as discussed in (2). When $\alpha=(-rT)/(1+r^{a.t.})$ the imputed wealth taxation system with tax payment rTS_0 in both states arises. The case $\alpha=0$ gives the relation (7) and is identical to the Auerbach (1991) taxation system to be discussed more thoroughly in section 5.

Observe that the choice of α and the magnitude of $(1+r^{a.t.})$ are independent. No matter what the value of α is, equation (8) is trivially true for any economically meaningful value of $(1+r^{a.t.})$, i.e. $r^{a.t.} > -1$[5]. The taxation of the riskless asset and the taxation of the risky assets may well be quite different, although naturally related. E.g., the riskless asset may be untaxed, i.e. $r^{a.t.} = r$, which leads to the following version of (8):

$$(1+r)S_0 = q\,[(1-\alpha)S^u + \alpha(1+r)S_0] + (1-q)\left[(1-\alpha)S^d + \alpha(1+r)S_0\right] \tag{9}$$

In this case α is essentially a tax rate in itself. $\alpha=0$ means no taxation at all, and $\alpha=1$ is again the »government takes all risk« mechanism or a tax rate of 100% acting on the net gain – in excess of an imputed riskless return rS_0 – as tax base. $\alpha=T$ is the mark-to-market taxation, but with untaxed interest earnings it is necessary – in order to maintain neutrality – to give allowance for a deduction of an imputed riskless return on the initial investment:

$$(1+r)S_0 = q\,[S^u - T(S^u - (1+r)S_0)] + (1-q)\left[S^d - T(S^d - (1+r)S_0)\right] \tag{10}$$

The findings for this simple one-period scenario are summarized in lemma 1.

Lemma 1 *The pricing relation (8) after tax for some investor*

- *subject to a linear and symmetric taxation*
- *with taxation given by* $\alpha \neq 1$ *and* $r^{a.t.} > -1$

is equivalent to the pricing relation (8) after tax for some other investor

- *subject to a linear and symmetric taxation*
- *with taxation given by* $\hat{\alpha} \neq 1$ *and* $\hat{r}^{a.t.} > -1$

Furthermore, the set of attainable claims after tax is the same for all such investors. With the exception of $\alpha=1$ *this set is identical to the set of attainable claims before tax.* ▮

Proof *See Appendix A.* ▮

3 The general discrete time model and mark-to-market valuation

One-period models are void of the problems with accrued capital gains, wash sale opportunities and timing options. The only concern for neutrality of a taxation system in a one-period model is valuation neutrality. However, the basic insight from the one-period model is necessary in order to study neutrality in a multi-period framework.

[5] Note that – at least as an experiment of thought – it is possible that a linear taxation system could magnify the investor's risk position compared to the pre-tax situation. The borderline value of α is precisely $-rT/(1+r^{a.t.})$, i.e. the imputed wealth tax system.

In a multi-period framework it is necessary to define taxation rules for accrued capital gains. In addition to valuation neutrality it is also necessary to examine whether the tax rules are holding period neutral.

In the rest of the paper we employ the standard setup for the discrete time model with a finite set of possible realizations[6]. I.e., we are given a probability space $(\Omega, \mathcal{F}, P)$ where Ω is a finite set of states: $\Omega \equiv \{1, 2, \ldots, M\}$ and a filtration $\{\mathcal{F}_t\}_{t=0}^{t=n}$ to be specified shortly.

In terms of this notation, $\{0, 1, 2, \ldots, n\}$ denotes the set of time indices, where trading in financial assets is allowed or where it is possible to either withdraw money from the portfolio in order to consume or to invest additional equity into the portfolio.

Without loss of generality we take $\mathcal{F}$ to be the discrete σ-algebra:

$$\mathcal{F}_n \subseteq \mathcal{F} = 2^{\Omega}, \qquad \mathcal{F}_0 = \{\emptyset, \Omega\}, \qquad P_\omega > 0 \;\; \forall \omega \in \Omega$$

The financial market has K traded assets. These assets are assumed not to pay dividends. Dividends are easily incorporated, but they merely contribute to a more complicated notation in the presentation of the basic arguments[7]. The prices at time t of these assets are denoted by the vector $S_t \in \mathbb{R}^K$. We take the filtration $\{\mathcal{F}_t\}_{t=0}^{t=n}$ as the »natural« filtration, i.e. the filtration generated by the K-dimensional price process S_t.

One of these assets is the bank account. This is denoted by B_t and develops as

$$B_{t+1} = (1 + r_t)B_t, \qquad B_0 \equiv 1 \tag{11}$$

where r_t is the one-period interest rate valid at time t for the period from time t until time $t+1$. The process r_t may well be stochastic, but it is required to be adapted to $\mathcal{F}_t$. Hence, the bank account is a predictable process, i.e. B_t is $\mathcal{F}_{t-1}$-measurable.

A trading strategy $(\theta^1, \theta^2, \ldots, \theta^n)$ is also a predictable process with values in $\mathbb{R}^K$. θ^j is the portfolio position held from time $j-1$ until time j. The exiting value at time $j-1$ is given by the inner product $\theta^j S_{j-1}$. The entering value at time j is similarly given by the inner product $\theta^j S_j$.

The following relation is a consequence of the "no arbitrage" pricing relations under any equivalent martingale measure:

$$\theta^1 S_0 = E^Q \left[\frac{\theta^n S_n}{B_n} + \sum_{j=1}^{n-1} \frac{(\theta^j - \theta^{j+1}) S_j}{B_j} \right] \tag{12}$$

Equation (12) expresses the valuation property that the value today of a trading strategy is the expected present value under Q of the realization value at the horizon time n, corrected for the net costs in present value terms of changing the portfolio at intermediate times $1, 2, \ldots, n-1$.

[6] The basics of the discrete time model is spelled out in numerous places. See e.g. Pliska (1997), chapter 3, for a good introduction or Jensen and Nielsen (1996) for a comprehensive development of the discrete time model with a finite sample space. None of the conclusions in the following depend in any essential manner on the assumption that the sample space is finite, cf. e.g. Kabanov and Kramkov (1995). However, some technicalities are simpler with this assumption with no loss of economic insight.

[7] Dividends are discussed shortly in section 6.

For a self-financing trading strategy, $\theta^j S_j = \theta^{j+1} S_j$, so the last term in (12) − the summation − disappears. This leads to the following relation for self-financing strategies:

$$0 = \sum_{j=1}^{n} E^Q \left[\frac{\theta^j S_j}{B_j} - \frac{\theta^j S_{j-1}}{B_{j-1}} \right] \tag{13}$$

Unless otherwise stated we assume in the following that the trading strategy $\theta^j,\ j=1,2,\ldots,n$, is self-financing on a before tax basis. In section 6 we treat the case where the portfolio is not self-financing, i.e. where the investor can make both withdrawals from the portfolio and capital injections into the portfolio at any point in time.

Consider the terms in (13) one by one. Due to the »no arbitrage« pricing relation these terms are individually zero, not only as unconditional expected value, but also when the expected values are conditioned on the filtration $\{\mathcal{F}_t\}_{t=0}^{t=n}$. Individually they fall into the category of the one-period model discussed in the binomial framework in section 2:

$$\frac{S_t}{B_t} = E^Q \left[\frac{S_{t+1}}{B_{t+1}} \middle| \mathcal{F}_t \right] \quad \Leftrightarrow \quad \theta^{t+1} S_t = E^Q \left[\theta^{t+1} S_{t+1} (1+r_t)^{-1} \middle| \mathcal{F}_t \right] \quad \forall \theta^{t+1} \in \mathcal{F}_t \quad \Leftrightarrow$$

$$E^Q \left[\theta^{t+1} \left(S_{t+1} - (1+r_t) S_t \right) \middle| \mathcal{F}_t \right] = 0 \quad \forall \theta^{t+1} \in \mathcal{F}_t \tag{14}$$

Hence, the procedure in (2)-(4) is applicable[8].

Let the tax rates applied be given as the predictable process $T_t,\ t=0,1,2,\ldots,n-1$[9]. Analogous to (11) the bank account corrected for taxation develops according to

$$B_{t+1}^{a.t.} = (1 + r_t(1-T_t)) B_t^{a.t.} \equiv (1 + r_t^{a.t.}) B_t^{a.t.}, \qquad B_0^{a.t.} \equiv 1 \tag{15}$$

By multiplying through by $1-T_t$ in any of the terms in (14) we get

$$0 = E^Q \left[\theta^{t+1} \left(S_{t+1} - S_t(1+r_t) \right) (1-T_t) \middle| \mathcal{F}_t \right] \Rightarrow \tag{16}$$

$$0 = E^Q \left[\theta^{t+1} \left(S_{t+1} - T_t(S_{t+1} - S_t) \right) - \theta^{t+1} S_t \left(1 + r_t^{a.t.} \right) \middle| \mathcal{F}_t \right] \Rightarrow \tag{17}$$

$$0 = E^Q \left[\frac{\theta^{t+1} \left(S_{t+1} - T_t(S_{t+1} - S_t) \right)}{B_{t+1}^{a.t.}} - \frac{\theta^{t+1} S_t}{B_t^{a.t.}} \middle| \mathcal{F}_t \right] \Rightarrow \tag{18}$$

$$\frac{\theta^{t+1} S_t}{B_t^{a.t.}} = E^Q \left[\frac{\theta^{t+1} \left(S_{t+1} - T_t(S_{t+1} - S_t) \right)}{B_{t+1}^{a.t.}} \middle| \mathcal{F}_t \right] \tag{19}$$

For a self-financing strategy this is also

$$\frac{\theta^t S_t}{B_t^{a.t.}} = E^Q \left[\frac{\theta^{t+1} \left(S_{t+1} - T_t(S_{t+1} - S_t) \right)}{B_{t+1}^{a.t.}} \middle| \mathcal{F}_t \right] \tag{20}$$

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[8] The procedure in lemma 1 is also applicable, but this section only deals with mark-to-market valuation.

[9] I.e., the tax rate for the period from time t to time $t+1$ is known at time t and is also known before any portfolio rebalancing decision is made at time t.

By backwards induction it is now straightforward to verify that

$$\frac{\theta^t S_t}{B_t^{a.t.}} = \frac{\theta^{t+1} S_t}{B_t^{a.t.}} = E^Q \left[\frac{\theta^n S_n}{B_n^{a.t.}} - \sum_{j=t}^{n-1} \frac{T_j \theta^{j+1}(S_{j+1} - S_j)}{B_{j+1}^{a.t.}} \,\middle|\, \mathcal{F}_t \right] \tag{21}$$

For $t=0$ the value $\theta^1 S_0$[10] in a before tax setting is compatible with identical values in an after tax setting, when taxes are paid on an accrual basis according to the mark-to-market value principle. The equivalent martingale measure remains the same; the effect of taxation shows up solely in the discounting factor and its cumulative effect, the value of the bank account $B_t^{a.t.}$ after tax.

Observe that this irrelevance proposition holds even if tax rates vary stochastically as a predictable process. Hence, political risk may exist as to the *level* of future tax rates. It is not necessary to know the tax rates in the future in order to value a payment stream correctly in this context. It is, however, crucial that the *structure* of taxation is known as being linear and symmetric. Furthermore, whether the interest rate process r_t is stochastic or not has no influence on this conclusion.

Again, the point of departure does not have to be a tax free investor. The point of departure might equally well have been *some* taxable investor for which a given set of prices does not produce arbitrage opportunities after tax. A slightly more tedious calculation, analogous to the one carried out in lemma 1, shows that these prices will then be compatible with *any* predictable schedule of tax rates for any other investor, as long as the tax schedule is linear and symmetric.

These results are stated in the following theorem.

Theorem 1 *For the general discrete time model the pricing relation (21) after tax for some investor*

- *subject to a linear and symmetric taxation and*
- *with tax rates following a predictable process T_t*

is equivalent to the pricing relation (21) after tax for some other investor

- *subject to a linear and symmetric taxation and*
- *with tax rates following a predictable process $\widehat{T}_t$.*

This is true for any equivalent martingale measure Q; hence the set of equivalent martingale measures is invariant to the particular predictable process that the tax rate may follow in a linear and symmetric mark-to-market valuation based taxation system. Furthermore, the set of attainable claims is the same for all such investors. ▮

Proof *See Appendix A.* ▮

[10] As a matter of taste one could define $\theta^0 \equiv \theta^1$ in order for the value at time 0 to be denoted $\theta^0 S_0$ instead of $\theta^1 S_0$.

4 Implementing a neutral realization based tax system

The typical objections to mark-to-market valuation based taxation are concerned with the liquidity needs of the investor and the informational requirements caused by repeated assessments of illiquid capital assets. For this reason there has been some interest in the literature to solve the liquidity problem by constructing *realization based* taxation systems that eliminate wash sale opportunities and induce holding period neutrality. Furthermore, a realization based taxation system that only needs to observe *net cash flows* in and out of the portfolio to form the tax base will solve the information problem.

Inspired by the structure of the solution (21) we define the time t *tax account* A_t^θ as the time t cumulative balance, cum interest, on the net taxes levied upon the portfolio strategy θ:

$$A_{t+1}^\theta = (1 + r_t^{a.t.})A_t^\theta + T_t\theta^{t+1}(S_{t+1} - S_t) \tag{22}$$

$$A_0^\theta \equiv 0 \tag{23}$$

The solution to this is

$$A_t^\theta = \sum_{j=0}^{t-1} T_j\theta^{j+1}(S_{j+1} - S_j)\frac{B_t^{a.t.}}{B_{j+1}^{a.t.}} \quad \Leftrightarrow \quad \frac{A_t^\theta}{B_t^{a.t.}} = \sum_{j=0}^{t-1} T_j \frac{\theta^{j+1}(S_{j+1} - S_j)}{B_{j+1}^{a.t.}} \tag{24}$$

By means of this tax account the pricing relation (21) can be rewritten as

$$\theta^t S_t - A_t^\theta = \theta^{t+1}S_t - A_t^\theta = E^Q\left[\frac{B_t^{a.t.}}{B_n^{a.t.}}\left(\theta^n S_n - A_n^\theta\right)\,\middle|\,\mathcal{F}_t\right] \quad \Leftrightarrow \tag{25}$$

$$\frac{\theta^t S_t - A_t^\theta}{B_t^{a.t.}} = \frac{\theta^{t+1}S_t - A_t^\theta}{B_t^{a.t.}} = E^Q\left[\frac{\theta^n S_n - A_n^\theta}{B_n^{a.t.}}\,\middle|\,\mathcal{F}_t\right] \tag{26}$$

This rewriting shows that any self-financing strategy θ^j before tax is in one-to-one correspondence with a self-financing strategy after tax. This self-financing strategy after tax may be denoted as $(\theta^j, -1)$, with "-1" referring to the position in the tax account A_j^θ.

The idea in the above account A_j^θ as a means to avoid liquidity problems dates back at least to Vickrey (1939)[11)]. The derivation shows – within the »no arbitrage« paradigm – that taxation in accordance with the mark-to-market valuation principle could be implemented in the form of a realization based taxation system in the sense that the investor pays the taxes accrued in the account A_j^θ whenever the position is realized. When reinvestment takes place the tax account is reset to zero value.

The realization based interpretation is also valid for the bank account itself. The valuation is independent of whether the bank account is interpreted as an asset with a constant market price and interest payments periodically paid and taxed or as a synthetic asset growing in value with

[11)] However, the primary purpose of Vickrey's suggestions were more inspired by a desire to even out the tax base and the tax payments over time than a desire to prove the neutrality of the mark-to-market valuation based taxation in a stringent manner. Progressive taxation schedules, carry forward provisions and similar aspects of the tax code are themselves arguments for even out procedures, and Vickrey cites earlier real life examples of implementations of such procedures.

periodical capital gains $r_t B_t$ to be taxed at realization. Given the above implementation in (26) the interpretation as a synthetic asset does not affect valuation and does not give rise to any timing options.

Imagine that a wash sale operation is attempted at time t. Then the investor's net proceeds is $\theta^t S_t - A_t^\theta$. If the investor borrows the balance A_t^θ in the bank account he will be able to reinvest $\theta^t S_t = \theta^{t+1} S_t$ in the originally planned, self-financing portfolio. If held until maturity n the net proceeds will be the gross result of the investment less (i) the time n value of the balance in the bank account and (ii) the future tax payments resulting from the reinvestment at time t:

$$\theta^n S_n - A_t^\theta \frac{B_n^{a.t.}}{B_t^{a.t.}} - \sum_{j=t}^{n-1} T_j \theta^{j+1}(S_{j+1} - S_j) \frac{B_n^{a.t.}}{B_{j+1}^{a.t.}} = \theta^n S_n - A_n^\theta \tag{27}$$

A similar argument can be used to show that this also holds for any stopping time strategy.

As a last remark, before moving to the analysis and characterization of linear taxation systems in general in the next section, observe that the realization principle and the resulting tax deferral incentives under a constant tax rate T are due to the fact that the tax account for the realization principle is governed by the analogous difference equation

$$A_{t+1}^\theta = A_t^\theta + T\theta^{t+1}(S_{t+1} - S_t) \tag{28}$$

$$A_0^\theta \equiv 0 \tag{29}$$

where the interest bearing element in the coefficient to A_t^θ has disappeared. However, when tax rates are allowed to vary stochastically the tax deferral incentives are less clear.

We will put the realization principle into the notation to be developed in the next section and use it as the premier example of a non-neutral taxation system.

5 Generalized linear taxation

In this section we ask and answer the following question: If the tax account is required to be a linear function of the sequence of valuations $\theta^t S_t$, what are the necessary and sufficient restrictions on the parameters in order for the taxation system to preserve the valuation neutrality and the holding period neutrality? The answer to this question will be given in an exhaustive form and the models referred elsewhere in this paper will appear as special cases together with the limitations necessary in order for them to work.

We start by specifying the value of the tax account in terms of »gross taxation rates« $K_{t,j}^0$ as follows:

$$A_t^\theta = \left(1 - K_{t,t}^0\right) \theta^t S_t - \sum_{j=0}^{t-1} K_{t,j}^0 \theta^j S_j \tag{30}$$

The superscript in $K_{t,j}^0$ refers to the starting date 0 as the last date where a transaction took place. We define $K_{0,0}^0 \equiv 1$ as the natural initial condition, i.e. the tax account has initial value zero.[12] In accordance with previous requirements it is natural to request that the tax laws are

[12] This means that the relations with the tax authorities are cleared at time 0 and that no relevant prehistory exists. This is generalized in section 6.

known at least one period in advance at the time where the last portfolio revision is allowed to take place. I.e. $K^0_{t,j} \in \mathcal{F}_{t-1}$ for $j=0,1,\dots,t$. Since $B^{a.t.}_t \in \mathcal{F}_{t-1}$ we also have that $K^0_{t,j}/B^{a.t.}_t \in \mathcal{F}_{t-1}$ for $j=0,1,\dots,t$.

Valuation neutrality requires that the after tax realization value at any date t, $t=0,1,2,\dots,n-1$, is equal to the time t value of the portfolio after tax at any future date up to and including the horizon n; i.e. for an appropriate martingale measure Q the following must hold for any pair of dates (t,m), $m \le n$, $t=0,1,2,\dots,m-1$:

$$\sum_{j=0}^{t} K^0_{t,j}\theta^j S_j = E^Q\left[\frac{\sum_{j=0}^{m} K^0_{m,j}\theta^j S_j}{(B^{a.t.}_m/B^{a.t.}_t)}\,\middle|\,\mathcal{F}_t\right] \qquad \Leftrightarrow \tag{31}$$

$$0 = E^Q\left[\sum_{j=0}^{t}\left(\frac{K^0_{m,j}}{B^{a.t.}_m} - \frac{K^0_{t,j}}{B^{a.t.}_t}\right)\theta^j S_j + \sum_{j=t+1}^{m}\left(\frac{K^0_{m,j}}{B^{a.t}_m}\right)\theta^j S_j\,\middle|\,\mathcal{F}_t\right] \tag{32}$$

In theorem 2 we state that for any »no arbitrage« pricing relation before tax and associated equivalent martingale measure Q the conditions in (33) are sufficient conditions for (32) to be fulfilled:

$$\frac{K^0_{m,j}}{B^{a.t}_m} = \frac{K^0_{t,j}}{B^{a.t}_t} \text{ for } j=0,1,\dots,t-1, \qquad K^0_{t,t} = B^{a.t.}_t \sum_{j=t}^{m} \frac{K^0_{m,j}}{B^{a.t.}_m}\frac{B_j}{B_t} \tag{33}$$

These conditions are also necessary provided that the stochastic dynamics of asset prices is sufficiently rich in the sense that non-trivial trading strategies exist; i.e. trading strategies such that $\theta^u S_u/B_u$ is never constant for any two adjacent date-events.[13] In terms of economic interpretation, the tax authorities or the legislators can choose a risk distribution mechanism between the government and the investor by choosing the tax parameters $(K^0_{n,0}, K^0_{n,1}, \dots, K^0_{n,n})$. Having done that the rest of the tax structure is determined by the development of the bank account after tax, i.e. determined by the way riskless returns are taxed over time.

The first part of the conditions in (33) implies that the tax consequences that would have occurred at time m, and ultimately at the horizon n, due to time periods already lapsed at time t are invariant to the choice of realization date; hence also invariant to timing options and attempted wash sales. The difference is solely due to the discounting effect of paying taxes later rather than sooner, whereas the risk sharing mechanism between the investor and the government is the same. Or, stated differently, the time 0 present value of the tax consequences when applying the value of the bank account after tax as a stochastic discount factor is independent of the choice of realization date. One consequence of this is that the first sum in (32) is eliminated, except for the term $j=t$. Another consequence is that $K^0_{m,j}/B^{a.t}_m \in \mathcal{F}_{t-1}$ for any $t>j$. Hence, $K^0_{m,j}/B^{a.t}_m \in \mathcal{F}_j$.

The second part of the conditions in (33) have on the r.h.s. the tax consequences, discounted to their time t present value, that would have occurred at time m, and ultimately at the horizon n, in case the portfolio value $\theta^t S_t$ had been rebalanced at time t and afterwards placed in the bank account for the remaining time periods. On the l.h.s. is the tax consequence, related to the portfolio value $\theta^t S_t$ at time t, of claiming the rebalancing as a premature realization. Hence, whether the investor claims that the portfolio is prematurely realized, with the tax consequences given by $(K^0_{t,0}, K^0_{t,1}, \dots, K^0_{t,t})$, or claims that it is rebalanced with future tax consequences given

[13] Alternatively stated the splitting index is always at least two. As a matter of fact this a very modest requirement.

by $(K^0_{m,0}, K^0_{m,1}, \ldots, K^0_{m,m})$ is a matter of indifference. Again, the time 0 present value of the tax consequences when applying the value of the bank account after tax as a stochastic discount factor is independent of the choice of realization date.

It is worth mentioning that these restrictions are independent of whether the market is complete or not, whether the short term interest rate is stochastic or not and whether the gross taxation rates are stochastic or not. The only requirement on the interest rate process and the gross taxation rates is that they are predictable.

Table 1 identifies the parameters $K^0_{n,j}$ for the particular cases referred to in this paper.

	$K^0_{n,j},\ j = 0, 1, 2, \ldots, n-1$	$K^0_{n,n}$
Mark-to-market	$\left[T_j - T_{j-1}(1 + r^{a.t.}_j)\right] \frac{B^{a.t.}_n}{B^{a.t.}_{j+1}};\ T_{-1} \equiv 0$	$1 - T_{n-1}$
Auerbach (1991)	0	$\frac{B^{a.t}_n}{B_n}$
Bradford (1995)/ Auerbach&Bradford (2001)	$1_{\{j=0\}} H B^{a.t}_n,\ H \in [0, 1]$	$(1 - H) \frac{B^{a.t}_n}{B_n}$
Brown (1948)	$1_{\{j=0\}} T B_n,\ B^{a.t.}_n = B_n$	$1 - T$
Imputed wealth tax	$-r_j T_j \frac{B^{a.t.}_n}{B^{a.t.}_{j+1}}$	1

Table 1: Parameterizations for five models

Among other things the second part of the conditions in (33) places strong restrictions on the possible stochastic variation in the tax parameters and their interplay with the possible stochastic variation in interest rates and tax rates: The sum of $\mathcal{F}_j$-measurable variables, $j = t, t+1, \ldots, m-1$, is a $\mathcal{F}_{t-1}$-measurable variable. The mark-to-market principle is a good example of this. Despite the possible stochastic feature of $K^0_{n,j}$ the sum of the r.h.s. in (33) is $(1 - T_{t-1})/B^{a.t.}_t$.

The conditions in (33) are mostly relative restrictions on the model parameters. However, for $t = 0$ the restriction $K^0_{0,0} \equiv 1$ is sufficient to determine the absolute values of the parameters; i.e.

$$1 = \sum_{j=0}^{n} \frac{K^0_{n,j} B_j}{B^{a.t.}_n} \quad \Leftrightarrow \quad B^{a.t.}_n = \sum_{j=0}^{n} K^0_{n,j} B_j \tag{34}$$

One way to read the restriction in (34) is that the result at the horizon of investing in one unit of the bank account at time 0 should be independent of whether the return is taxed period by period as interest income or taxed by realization at the horizon as capital gain.

Next consider a wash sale operation at time t. We assume that the self-financing portfolio policy is reestablished by taking a short position in the bank account to pay the tax liability A^θ_t and restart the linear tax system at time t with new coefficients $K^t_{n,j}$. Then – analogous to (31)–(32)

− holding period neutrality is guaranteed whenever

$$\theta^t S_t - A_t^\theta = E^Q\left[\frac{\sum_{j=t}^s K_{s,j}^t \theta^j S_j - A_t^\theta\left(B_s^{a.t.}/B_t^{a.t.}\right)}{(B_s^{a.t.}/B_t^{a.t.})}\,\middle|\,\mathcal{F}_t\right],\quad s=t+1,\ldots,n \quad\Leftrightarrow \tag{35}$$

$$\theta^t S_t = E^Q\left[\frac{\sum_{j=t}^s K_{s,j}^t \theta^j S_j}{(B_s^{a.t.}/B_t^{a.t.})}\,\middle|\,\mathcal{F}_t\right],\quad s=t+1,\ldots,n \tag{36}$$

The necessary and sufficient restrictions on the revised coefficients are identical to the restrictions on the original ones. Since the starting date is now t we have − analogous to (33) − the conditions

$$\frac{K_{n,j}^t}{(B_n^{a.t}/B_t^{a.t.})} = \frac{K_{s,j}^t}{(B_s^{a.t}/B_t^{a.t.})} \quad\Leftrightarrow\quad \frac{K_{n,j}^t}{B_n^{a.t}} = \frac{K_{s,j}^t}{B_s^{a.t}} \quad \text{for } j=t,t+1,\ldots,s-1 \tag{37}$$

$$\frac{K_{s,s}^t}{(B_s^{a.t.}/B_t^{a.t.})} = \sum_{j=s}^n \frac{K_{n,j}^t}{(B_n^{a.t.}/B_t^{a.t.})}\frac{B_j/B_t}{B_s/B_t} \quad\Leftrightarrow\quad K_{s,s}^t = B_s^{a.t.}\sum_{j=s}^n \frac{K_{n,j}^t}{B_n^{a.t.}}\frac{B_j}{B_s} \tag{38}$$

According to (37)-(38) one sufficient condition guaranteeing holding period neutrality is to continue with the original model parameters for taxation; however, the discounting must be changed so that future values are discounted back to time t. This provides the necessary scaling so that the revised parameters satisfy the restriction

$$1 = \sum_{j=t}^n K_{n,j}^t \frac{B_t^{a.t.}}{B_n^{a.t.}}\frac{B_j}{B_t} \tag{39}$$

Since the tax account is reset to value zero it is also possible to "start from scratch" and change to any another neutral system upon realization[14)].

We now state the main results of the generalized linear taxation system in theorem 2.

Theorem 2 *Linear and symmetric neutral taxation schedules in the discrete time "no arbitrage" pricing model are exhaustively characterized as follows.*

1. *For any given "no arbitrage" equilibrium before tax and related equivalent measure Q the parameter restrictions for any pair of dates (t,m), $t=0,1,2,\ldots,m-1$, $m\le n$:*

$$\frac{K_{m,j}^0}{B_m^{a.t}} = \frac{K_{t,j}^0}{B_t^{a.t}} \quad \textit{for } j=0,1,\ldots,t-1, \qquad \frac{K_{t,t}^0}{B_t^{a.t.}} = \sum_{j=t}^m \frac{K_{m,j}^0}{B_m^{a.t.}}\frac{B_j}{B_t}, \qquad K_{0,0}^0 = 1 \tag{40}$$

   are sufficient to produce the analogous "no arbitrage" equilibrium after tax:

$$\frac{1}{B_t^{a.t.}}\sum_{j=0}^t K_{t,j}^0 \theta^j S_j = E^Q\left[\frac{\sum_{j=0}^m K_{m,j}^0 \theta^j S_j}{B_m^{a.t.}}\,\middle|\,\mathcal{F}_t\right] \tag{41}$$

[14)] Section 6 outlines one way to implement this recalculation of coefficients that is particularly well suited for partial realizations.

2. *Assume that the stochastic dynamics of asset prices is sufficiently rich in the sense that non-trivial self-financing trading strategies exist, i.e. self-financing trading strategies such that $\theta^u S_u/B_u$ is never constant across adjacent date-events. Then valuation neutrality for any given taxable investor in the sense that (41) is fulfilled for some equivalent martingale measure Q and any pair of dates (t,m), $t=0,1,2,\ldots,m-1$, $m \leq n$, requires the parameter conditions in (40) as necessary conditions. If, furthermore, $K_{tt} \neq 0\ \forall t$ then valuation neutrality for some taxable investor with some equivalent martingale measure Q implies valuation neutrality for any other investor with the same equivalent martingale measure, but other taxation parameters satisfying (40).*

3. *Define the $\mathcal{F}_j$-measurable variables ψ_j and the $\mathcal{F}_{t-1}$-measurable variables ϕ_t as follows:*

$$\psi_j \equiv \frac{K^0_{n,j}}{B^{a.t.}_n},\ j=0,1,\ldots,n-1, \quad \phi_t \equiv \frac{K^0_{t,t}}{B^{a.t.}_t},\ t=1,2,\ldots,n, \quad \phi_0 \equiv 1 \tag{42}$$

Then the following relations are sufficient conditions for (40) to hold:

$$\phi_t B_t = \phi_n B_n + \sum_{j=t}^{n-1} \psi_j B_j, \quad t=0,1,2,\ldots,n-1 \tag{43}$$

$$\psi_j = \frac{K^0_{t,j}}{B^{a.t.}_t},\ j=1,2,\ldots,t-1 \quad \text{and } t=1,2,3,\ldots,n-1 \tag{44}$$

$$\psi_j = \phi_j-(1+r_j)\phi_{j+1}; \quad \phi_0 \equiv 1, \quad \phi_{n+1} \equiv 0 \tag{45}$$

4. *The following conditions are sufficient – and under the same assumptions as in 2. necessary – to ensure holding period neutrality, i.e. the elimination of any wash sale opportunity at time t:*

$$\frac{K^t_{n,j}}{(B^{a.t}_n/B^{a.t.}_t)} = \frac{K^t_{s,j}}{(B^{a.t}_s/B^{a.t.}_t)} \quad \Leftrightarrow \quad \frac{K^t_{n,j}}{B^{a.t}_n} = \frac{K^t_{s,j}}{B^{a.t}_s} \qquad \text{for } j=t,t+1,\ldots,s-1 \tag{46}$$

$$\frac{K^t_{s,s}}{(B^{a.t.}_s/B^{a.t.}_t)} = \sum_{j=s}^{n} \frac{K^t_{n,j}}{(B^{a.t.}_n/B^{a.t.}_t)} \frac{B_j/B_t}{B_s/B_t} \quad \Leftrightarrow \quad \frac{K^t_{s,s}}{B^{a.t.}_s} = \sum_{j=s}^{n} \frac{K^t_{n,j}}{B^{a.t.}_n}\frac{B_j}{B_s} \tag{47}$$

One possible choice of tax parameters $K^t_{n,j}$ following a wash sale at time t is to set these parameters equal to their equivalent values in an identical model with time horizon $n-j$ and change the reference point for discounting to time t.

5. *Any convex combination of two linear and symmetric neutral taxation schedules is again a linear and symmetric taxation schedule.*

6. *The set of attainable claims after tax is the same for all investors for which $\phi_t \neq 0\ \forall t$, and this set is identical to the set of attainable claims before tax.* ∎

Proof *Part of the proof has already been sketched above. The completion of the proof is found in Appendix A.* ∎

From relation (45) and straightforward manipulations we have the following corollary:

Corollary 1 *A neutral taxation system is given by its diagonal elements K^0_{tt} and the wealth dynamics for an unrealized portfolio position is given by*

$$(\theta^{t+1}S_{t+1} - A^\theta_{t+1}) - (1 + r_t(1 - T_t))\left(\theta^t S_t - A^\theta_t\right) =$$
$$K^0_{t+1,t+1}\left[\theta^{t+1}S_{t+1} - (1 + r_t)\theta^t S_t\right] \qquad (48)$$

The tax account develops according to

$$A^\theta_{t+1} = (1 + r_t(1 - T_t))A^\theta_t + r_t T_t \theta^t S_t + (1 - K^0_{t+1,t+1})(\theta^{t+1}S_{t+1} - (1 + r_t)\theta^t S_t) \qquad (49)$$

∎

According to corollary 1 any neutral taxation system decomposes into two elements:

1. an imputed wealth tax at tax rate T_t, which can be interpreted as a tax on the change in wealth due to "passage of time", $r_t\theta^t S_t$
2. a tax on the risky part of the pre-tax earnings, i.e. on pre-tax earnings net of the wealth change due to "passage of time", with tax rate $1-K^0_{tt}$.

Hence, the diagonal elements $1-K^0_{tt}$ have the role of tax rates applicable to the risky income component.

Example 2: Parameter identification for well-known models

For the mark-to-market valuation we have

$$\phi_t \equiv \frac{1 - T_{t-1}}{B^{a.t.}_t}, \quad \phi_{t+1} \equiv \frac{1 - T_t}{B^{a.t.}_{t+1}} \qquad (50)$$

$$\begin{aligned}\psi_t &= \phi_t - \phi_{t+1}(1 + r_t) \\ &= \frac{1 - T_{t-1}}{B^{a.t.}_t} - \frac{1 - T_t}{B^{a.t.}_{t+1}}(1 + r_t) \qquad (51)\\ &= \frac{1 - T_{t-1}}{B^{a.t.}_t} - \frac{1 + r_t(1 - T_t) - T_t}{B^{a.t.}_{t+1}} \qquad (52)\\ &= \frac{T_t}{B^{a.t.}_{t+1}} - \frac{T_{t-1}}{B^{a.t.}_t} \qquad (53)\end{aligned}$$

For fixed T and a constant interest rate this becomes

$$\psi_t = -\frac{Tr^{a.t.}}{(1 + r^{a.t.})^{t+1}} \qquad (54)$$

Observe that for fixed T the mark-to-market valuation principle is actually a convex combination of two other linear taxation systems. The coefficients $K^0_{n,j}$ in each period arise as the imputed wealth tax weighted by $(1-T)$ and the »government takes all risk« tax weighted by T.

Mark-to-market value taxation is also characterized by taxing the two income components in Corollary 1 in the same way with identical tax rates.

Auerbach's ≫retrospective capital gains tax≪ model, cf. Auerbach (1991), has

$$\phi_t = 1/B_t \quad \text{and} \quad \psi_t = 0 \tag{55}$$

The "diagonal element" tends to zero for long time horizons. This means that the taxation system approaches a pure imputed wealth taxation in the long run.[15)]

Auerbach's model has been generalized in Bradford (1995) and Auerbach and Bradford (2001). Bradford (1995) and Auerbach and Bradford (2001) are identical models in the sense that the present value of the taxes levied upon an investment are identical, but the implementation is different. Whereas the taxation system in Bradford (1995) taxes capital gains at a so-called "gains recognition date", irrespective of realization decisions and cash flows, Auerbach and Bradford (2001) is the true cash flow taxation version of Bradford (1995). This generalized model is similar in structure to the *pure cash flow tax* or *Brown tax* proposed in Brown (1948), except that it is required that the bank account after tax grows with the after tax rate of interest instead of the before tax rate of interest[16)].

Auerbach and Bradford's generalized cash flow taxation model (Auerbach and Bradford (2001)) has

$$\phi_t = \frac{1-H}{B_t}, \quad \psi_0 = H \quad \text{and} \quad \psi_t = 0 \text{ for } j = 1, 2, \ldots, n-1 \tag{56}$$

Auerbach and Bradford's generalized cash flow taxation model is also a convex combination of two other schedules. One of them is the Auerbach (1991) system which is the ≫corner solution≪ with $H = 0$. The other is the ≫government takes all risk≪ taxation system, which is the ≫corner solution≪ with $H = 1$.

The mark-to-market taxation principle would turn into the Auerbach and Bradford (2001) taxation system if the tax rate T_t was growing with the same growth rate as the bank account after tax. Under such circumstances $K^0_{n,j} = 0$ for $j = 1, 2, \ldots, n-1$ and $K^0_{n,0} = T_0(B^{a.t.}_n / B^{a.t.}_1)$. The taxation at the horizon must then be

$$K^0_{n,n} = \frac{B^{a.t.}_n}{B_n}\left(1 - \frac{T_0}{B^{a.t.}_1}\right)$$

Hence, $H = T_0 / B^{a.t.}_1$.

The Brown tax is equivalent to the Auerbach and Bradford (2001) cash flow taxation system with $H = T$ except that interest earnings are left untaxed. ∎

Example 3: A neutral ≫averaging≪ tax system

Consider the following parameters for a neutral taxation system:

$$\phi_0 = 1, \qquad \phi_j = \frac{1-(j-1)/n}{B_j}, \qquad \phi_{n+1} = 0, \qquad \psi_j = \frac{1}{nB_j} \tag{57}$$

[15)] This is different from the original exposition in Auerbach (1991), which was formulated in continuous time. In continuous time the retrospective capital gains taxation system is indistinguishable from the imputed wealth taxation system.

[16)] The original work of Brown (1948) dealt with investment theory and the design of neutral tax depreciation rules. The tax system was characterized as taxing the return on real investment with a tax base solely based on "non financial" cash flows.

This is one example of a generalized version of the Auerbach (1991) taxation system, where the tax base is averaged out over the entire period to depend on all the portfolio values $\theta^j S_j$, $j=1,2,\ldots,n$ instead of being dependent solely on the value observed at the horizon. It is easily checked that the system is neutral by plugging into, e.g., the conditions in (43)-(45).

If the position is realized at time t the investor's after tax revenue is

$$\sum_{j=0}^{t} K^0_{t,j}\theta^j S_j = \frac{1}{n}\sum_{j=1}^{t-1}\frac{B^{a.t.}_t}{B_j}\theta^j S_j + \left(1-\frac{t-1}{n}\right)\frac{B^{a.t.}_t}{B_t}\theta^t S_t \tag{58}$$

The weight $1/n$ is applied to the taxation of each of the terms $\theta^j S_j$, $j=1,2,\ldots,t-1$, in accordance with the Auerbach (1991) taxation system which leaves the investor with $(B^{a.t.}_j/B_j)\theta^j S_j$ from the j'th term. This result is carried forward to time t with the bank account after tax, $B^{a.t.}_t/B^{a.t.}_j$. The remaining weight, $(1-(t-1)/n)$, is applied to the taxation of the term $\theta^t S_t$. ∎

Example 4: Readjustment due to wash sale

A neutral taxation principle is required to remain unaffected by wash sale attempts. For the mark-to-market valuation we have

$$K^0_{n,t} = B^{a.t.}_n\left[\frac{T_t}{B^{a.t.}_{t+1}}-\frac{T_{t-1}}{B^{a.t.}_t}\right], \quad K^0_{t,t} = 1-T_{t-1} \tag{59}$$

$$K^t_{n,s} = \frac{B^{a.t.}_n}{B^{a.t.}_t}\left[\frac{T_s}{\left(B^{a.t.}_{s+1}/B^{a.t.}_t\right)}-\frac{T_{s-1}}{\left(B^{a.t.}_s/B^{a.t.}_t\right)}\right] = B^{a.t.}_n\left[\frac{T_s}{B^{a.t.}_{s+1}}-\frac{T_{s-1}}{B^{a.t.}_s}\right] \tag{60}$$

$$K^t_{s,s} = 1-T_{s-1} \tag{61}$$

Hence, the mark-to-market value system – as expected – is unable to observe any wash sale. Any taxation effect up to and including the wash sale date has already been activated.

For Auerbach's »retrospective capital gains tax« model we have

$$K^0_{n,t}=0,\ t\neq n, \qquad K^0_{t,t}=\frac{B^{a.t.}_t}{B_t} \tag{62}$$

$$K^t_{n,s}=0,\ s\neq n, \qquad K^t_{s,s}=\frac{\left(B^{a.t.}_s/B^{a.t.}_t\right)}{\left(B_s/B_t\right)} \tag{63}$$

The Auerbach (1991) system taxes the investor by retaining a certain fraction of the wealth. This fraction depends on the holding period in a monotonous way. When a certain fraction is captured at a realization date $t<n$, the remaining holding period is shortened, so the fraction to be captured at the end of the horizon – out of the remaining wealth – is smaller. This makes the sequence of taxation dates irrelevant as shown in (64):

$$\frac{B^{a.t.}_n}{B_n} = \left(\frac{B^{a.t.}_t}{B_t}\right)\cdot\left(\frac{B^{a.t.}_n/B^{a.t.}_t}{B_n/B_t}\right) \tag{64}$$

Auerbach&Bradford's generalized cash flow taxation model has

$$K^0_{n,t} = 1_{\{t=0\}}HB^{a.t.}_n, \quad K^0_{t,t} = (1-H)\frac{B^{a.t.}_t}{B_t} \tag{65}$$

Generally speaking the system could be restarted with any new value of the constant H. However, following the logic of the model as described in Auerbach and Bradford (2001), the only possible continuation with the "same structure" is to switch to the Auerbach (1991) system after a realization. So the tax rebate due to acquisition costs is given precisely once at the time of the first realization:

$$K_{n,s}^t = 0,\ s \neq n, \quad K_{s,s}^t = \frac{B_s^{a.t.}/B_t^{a.t.}}{B_s/B_t} \tag{66}$$

∎

<u>Example 5: The realization principle and non-neutrality</u>

The realization principle is characterized by the parameter values:

$$K_{nn}^0 = 1 - T, \qquad K_{n0}^0 = T, \qquad K_{nj}^0 = 0 \text{ for } j = 1, 2, \ldots, n-1 \tag{67}$$

It is *not* a neutral taxation system. This can be verified by checking, e.g., condition (34):

$$\frac{K_{nn}^0}{B_n^{a.t.}} B_n + \frac{K_{n0}^0}{B_n^{a.t.}} = \frac{(1-T)B_n + T}{B_n^{a.t.}} \stackrel{?}{=} 1 \tag{68}$$

For $n = 1$ this is fulfilled – there is no deferral problem in a one-period context. However, for $n > 1$ we have that[17)]

$$\frac{(1-T)B_n + T}{B_n^{a.t.}} > 1 \tag{69}$$

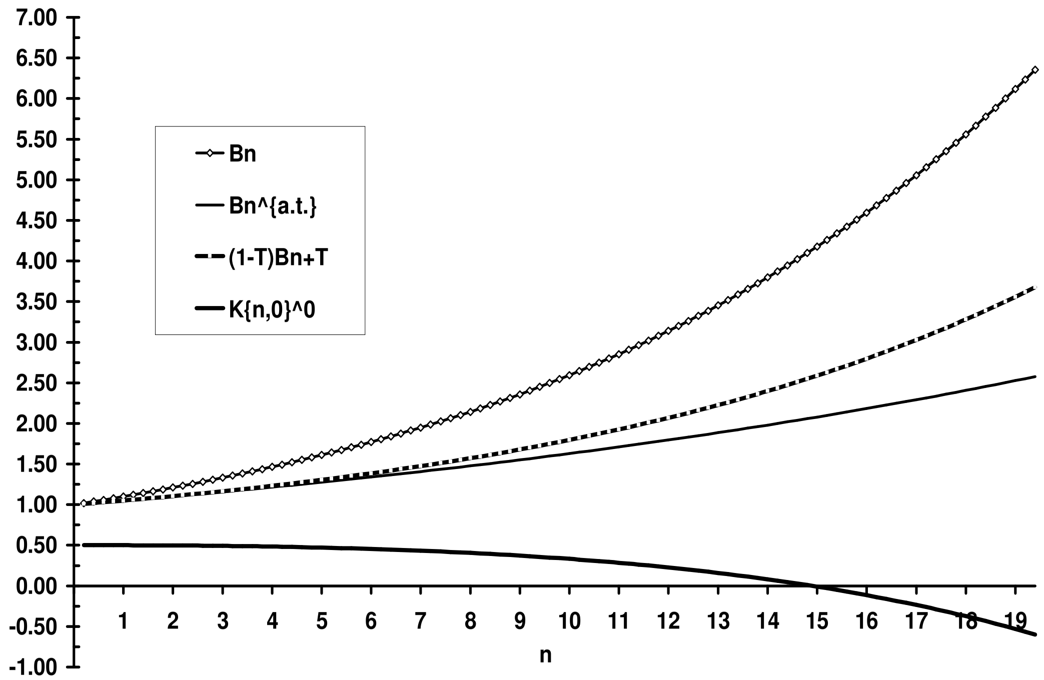

Figure 1: The value of $K_{n,n}^0$ in a neutral realisation based taxation system. Parameter values: $r = 10\%$ and $T = 50\%$.

17) This follows, e.g., from Jensen's inequality applied to the convex function $x \to x^n$ and the arguments $1 + r$ and 1, respectively.

Solving for the "correct" value of K^0_{n0} in order to make the realization principle neutral shows that the tax value of deducing the initial investment from the tax basis must be less than T:

$$K^0_{n0} = B^{a.t.}_n - (1-T)B_n < T \tag{70}$$

It is diminishing with the length of the time horizon and for sufficiently large values of n, K^0_{n0} even becomes negative! With parameter values $r = 10\%$ and $T = 50\%$ this happens for n close to 15.

Figure 1 shows four curves. The upper curve is the value of the bank account before tax, and the third curve from above is the value of the bank account after tax. The curve between these two shows the actual end result from investing 1 in the bank account and having the returns taxed on a realization basis. The lower curve is the value of $K^0_{n,0}$ that corresponds to tax neutrality. ▮

6 Multiple investments, withdrawals and neutral taxation

So far we have assumed that the portfolios were self-financing. This is an important simplification and not only for the theoretical developments. Real life tax codes have different – sometimes slightly complicated – rules for how to account for assets that are identical except for the acquisition date; e.g. stocks in the same company bought and sold at different times. Typical examples include (i) the assets sold are precisely identified by choice of the investor (ii) the assets sold are identified by use of the FIFO principle, i.e. assets are identified in chronological order (iii) the assets sold are identified by use of the LIFO principle, i.e. assets are identified in reverse chronological order and (iv) the assets sold are *not* identified, but an average acquisition price for the entire position is calculated – and recalculated upon any additional purchase – and used as the basis for calculating taxable capital gains and losses.

Within the class of generalized linear taxation systems it is, of course, a possibility to isolate any new injection of funds as an independent investment on its own. However, this would require a new set of weights to be initiated for every new injection because the holding period must be identified individually; and rules, similar to the ones above, must be determined for withdrawals.

It turns out that there is a simple recipe for handling the tax treatment of portfolios and portfolio changes that are not self-financing. In case of a withdrawal from the portfolio at time t the withdrawn part is simply treated as a realization in accordance with the coefficients $K^0_{t,j}$, $j = 0, 1, \dots, t$. When a fraction x of the portfolio value is realized at time t, the tax calculations for future realizations are reduced by the factor $1-x$. On the other hand if at time t an additional purchase is made through a capital injection into the portfolio a correction to the tax account A^θ_t is needed. We will determine this correction by use of a variational analysis, taking an existing self-financing portfolio strategy as the point of departure.

Let the capital injection be denoted by $(\Delta\theta)^t S_t$, where the portfolio change due to the capital injection is identified by the symbol Δ. In the time periods after t this capital injection gives rise to changes $(\Delta\theta)^j S_j$, $j = t+1, \dots, n$, in the portfolio values. If one ignores the distinction between "existing portfolio" and "capital injection" after time t the condition (31) for neutrality boils down to

$$K^0_{t,t}(\Delta\theta)^t S_t = E^Q\left[\frac{\sum_{j=t}^m K^0_{m,j}(\Delta\theta)^j S_j}{(B^{a.t.}_m / B^{a.t.}_t)}\,\middle|\,\mathcal{F}_t\right] \tag{71}$$

Since the investor is investing $(\Delta\theta)^t S_t$, which must be the value of the variational portfolio changes after tax, the investor can be fully compensated for not having the holding period reset by crediting the tax account at time t with the amount $(1 - K^0_{t,t})(\Delta\theta)^t S_t$.

In appendix C we show the result of this procedure for the Auerbach (1991) taxation system as well as for the mark-to-market valuation system and verify that the procedure is equivalent in terms of valuation to resetting the holding period. The Brown tax is self-explanatory in this respect[18].

Now consider the case of a withdrawal, e.g. due to a dividend payment. If dividends are paid out and not reinvested they are taxed precisely as any other realization of a part of the portfolio. The net proceeds after tax from a withdrawal at time t making up the fraction x of the portfolio value is

$$x \sum_{j=0}^{t} K^0_{t,j} \theta^j S_j \tag{72}$$

and the taxes to be paid on future realizations are scaled down by the factor $1-x$ for time indices $j = 0, 1, 2, \ldots, t$. If held until the horizon n the net proceeds after taxes, expressed in terms of the original portfolio strategy θ^j, becomes

$$(1 - x) \sum_{j=0}^{n} K^0_{n,j} \theta^j S_j \tag{73}$$

Adding to this the proceeds from time t, carried forward to time n, we end up with a total time n proceeds as follows:

$$\sum_{j=0}^{n} K^0_{n,j} \theta^j S_j + x \left[\frac{B^{a.t.}_n}{B^{a.t.}_t} \left(\sum_{j=0}^{t} K^0_{t,j} \theta^j S_j \right) - \sum_{j=0}^{n} K^0_{n,j} \theta^j S_j \right] \tag{74}$$

After appropriate reductions it is a simple calculation to show that the last part of (74) has zero time t-value, whenever Q is an equivalent martingale measure on a before tax basis. The requirements for this to be true are very modest, cf. statement 2 in theorem 2. The details are given in Appendix C. This term is not identically zero due to the implicit riskless carry forward of the taxes paid at time t, but its time t value is zero.

Alternatively, if the tax account is updated in accordance with the recursive formula in (49) it is straightforward to account for a payment of the fraction x in case of a withdrawal at time t. Everything in (49) is scaled down by the factor $1-x$ and the recursive updating mechanism can just continue at a reduced level.

Assume alternatively that the dividends are reinvested and that the net tax bill to be paid at time t is borrowed in the bank account. According to the scheme outlined the reinvestment would lead to an immediate tax benefit of the magnitude $x(1 - K^0_{t,t})\theta^t S_t$; hence, the net taxes paid at time t become

$$-\sum_{j=0}^{t-1} K^0_{t,j} x \theta^j S_j \tag{75}$$

[18] For the Bradford (1995) taxation model this procedure does not work – or, rather, it would be reduced to an equivalent of the Auerbach (1991) system. For the Bradford (1995) system the key feature can only be maintained by tracking each separate capital injection individually and with its own clock.

Repeating the carry forward argument the taxes to be paid at time n under this reinvestment scenario become

$$\left((1-K^0_{n,n})\theta^n S_n - \sum_{j=t}^{n-1} K^0_{n,j}\theta^j S_j\right) - (1-x)\sum_{j=0}^{t-1} K^0_{n,j}\theta^j S_j \tag{76}$$

Subtracting the carry forward value of the taxes paid at time t, cf. (75), leads to the net result at time n:

$$\left(K^0_{n,n}\theta^n S_n + \sum_{j=t}^{n-1} K^0_{n,j}\theta^j S_j\right) + (1-x)\sum_{j=0}^{t-1} K^0_{n,j}\theta^j S_j + \frac{B^{a.t.}_n}{B^{a.t.}_t}\sum_{j=0}^{t-1} K^0_{t,j} x\theta^j S_j = \tag{77}$$

$$\sum_{j=0}^{n} K^0_{n,j}\theta^j S_j - x\left(\sum_{j=0}^{t-1}\left(K^0_{n,j} - \frac{B^{a.t.}_n}{B^{a.t.}_t}K^0_{t,j}\right)\theta^j S_j\right) \tag{78}$$

Because of the relations (33) each individual term in the latter sum is zero. Hence, these derivations prove the holding period neutrality of the tax mechanism proposed for partial realizations – including wash sales – and dividend payments.

7 Summary and conclusion

The concept of a neutral taxation system is a normative benchmark with which other taxation systems may be compared and the severeness of the deviations measured. A neutral taxation system has the property that all investors, irrespective of their individual tax situation, agree on the market prices and that attempts to exploit wash sale opportunities in order to generate tax deferrals are ruled out.

This paper investigates the consequences of the "no arbitrage" assumption applied to after tax values. We provide necessary and sufficient conditions that characterize in an exhaustive manner when a linear and symmetric taxation system is a neutral taxation system. The paper derives the results for the one-period binomial model in section 2 and the results for the mark-to-market valuation principle in the general discrete time model in section 3. Section 4 introduces the concept of the tax account where tax liabilities can be kept track of in an interest bearing manner. This idea is crucial for the structure of the analysis in section 5 and for theorem 2, which is the main result of the paper. This theorem encompasses existing suggestions in the literature. Besides the mark-to-market valuation principle this is the Brown (1948) tax, the imputed wealth tax, the Auerbach (1991) retrospective capital gains tax and the Bradford (1995) and Auerbach and Bradford (2001) generalized cash flow taxation systems. The analysis in this paper also points out in what sense and to what extent interest rates and tax rates can be stochastic.

The results in the paper, and in particular in theorem 2, do not depend in any way on equilibrium model properties. They are directly in line with the ≫no arbitrage≪ paradigm of mathematical finance which builds on a minimum requirement for an asset pricing model, namely that arbitrage opportunities do not exist.

For the *general version* of theorem 2 we need to have a given finite time horizon. This is a weakness when one thinks in terms of implementation along the lines of the tax rules discussed here. Fortunately, this requirement is not needed for any of the specific examples given in table 1, where the mechanisms can be extended to any horizon.

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A Proofs

Proof (Lemma 1) *In equation (8) the factor* $(1+r^{a.t.})$ *can be changed to any other factor* $(1+\widehat{r}^{a.t.})$ *by multiplication. This is so independent of the value of* α. *The pricing equation (8) is valid for the riskless asset under all circumstances, since* $S^u = S^d = 1+r,\ S_0 = 1$ *trivially satisfies (8).*

The following calculation follows immediately from (8) when $\alpha \neq 1$ *as well as* $\widehat{\alpha} \neq 1$ *:*

$$
\begin{aligned}
\left(1+r^{a.t.}\right) S_0 &= q\frac{1+r^{a.t.}}{1+r}(1-\alpha)S^u + (1-q)\frac{1+r^{a.t.}}{1+r}(1-\alpha)S^d + \\
&\quad \alpha\left(1+r^{a.t.}\right) S_0 && \Leftrightarrow \\
(1+r^{a.t.})S_0 &= q\frac{1+r^{a.t.}}{1+r}S^u + (1-q)\frac{1+r^{a.t.}}{1+r}S^d && \Leftrightarrow \\
(1+r^{a.t.})(1-\widehat{\alpha})S_0 &= q\frac{1+r^{a.t.}}{1+r}(1-\widehat{\alpha})S^u + (1-q)\frac{1+r^{a.t.}}{1+r}(1-\widehat{\alpha})S^d && \Leftrightarrow \\
(1+r^{a.t.})S_0 &= q\frac{1+r^{a.t.}}{1+r}(1-\widehat{\alpha})S^u + (1-q)\frac{1+r^{a.t.}}{1+r}(1-\widehat{\alpha})S^d + \\
&\quad \widehat{\alpha}(1+r^{a.t.})S_0 && \text{(A.1)}
\end{aligned}
$$

The set of attainable claims is the subspace spanned by the two vectors (S_u, S_d) *and* $(1,1)$[19]*:*

$$
\mathcal{M} \equiv \left\{ \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \,\middle|\, \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \theta_1 \begin{pmatrix} S_u \\ S_d \end{pmatrix} + \theta_0 \begin{pmatrix} 1 \\ 1 \end{pmatrix},\ (\theta_0, \theta_1) \in I\!R^2 \right\} \qquad \text{(A.2)}
$$

When $\alpha \neq 1$ *this is obviously the same subspace as the one spanned by the vectors*

$$
\left[(1-\alpha)\begin{pmatrix} S_u \\ S_d \end{pmatrix} + (1+r)\alpha \begin{pmatrix} S_0 \\ S_0 \end{pmatrix}\right] \quad \text{and} \quad \begin{pmatrix} S_0 \\ S_0 \end{pmatrix}
$$

Each asset produces an after tax value proportional to its before tax value. However, when $\alpha = 1$ *the only after tax values attainable are the values proportional to the vector* $(1,1)$.

Hence, the set of attainable claims is invariant to a linear and symmetric taxation schedule for $\alpha \neq 1$. ∎

Proof (Theorem 1) *Self-evidently, none of the tax rates are allowed to be 1. The following sequence of calculations immediately follows from (20):*

$$
\begin{aligned}
(1+r_t(1-T_t))S_t &= E^Q\left[S_{t+1} - T_t(S_{t+1}-S_t) \,\middle|\, \mathcal{F}_t\right] && \Leftrightarrow \\
(1+r_t)(1-T_t)S_t &= E^Q\left[S_{t+1}(1-T_t) \,\middle|\, \mathcal{F}_t\right] && \Leftrightarrow \\
(1+r_t)(1-\widehat{T}_t)S_t &= E^Q\left[S_{t+1}(1-\widehat{T}_t) \,\middle|\, \mathcal{F}_t\right] && \Leftrightarrow \\
(1+r_t(1-\widehat{T}_t))S_t &= E^Q\left[S_{t+1} - \widehat{T}_t(S_{t+1}-S_t) \,\middle|\, \mathcal{F}_t\right] && \text{(A.3)}
\end{aligned}
$$

[19] This subspace is trivially $I\!R^2$ in this simple setup, but for completeness a formal proof is given.

Hence, this shows that the before tax pricing relation (14) can be deduced from the after tax pricing relation (20) and that any equivalent martingale measure for some investor with predictable tax rate process T_t is also an equivalent martingale measure for any other investor with predictable tax rate process $\widehat{T}_t$.

The set of attainable claims is determined by the ability to revise the portfolio at each node in the tree. The set of possible portfolio values at time $t+1$ is the subspace $\mathcal{M}_t$ spanned by the vectors S_{t+1}; i.e.

$$\mathcal{M}_t = \left\{ X \in I\!R^M \;\middle|\; X = \theta S_{t+1}, \quad \theta \in I\!R^N \right\} \tag{A.4}$$

In order to see that this is the same subspace as the subspace spanned by the after tax values one must observe that the filtration $\{\mathcal{F}_t\}_{t=0}^{t=n}$ also generates a partition of Ω into atoms $A_i \in \mathcal{F}_t$. Atoms are the smallest subsets in $\mathcal{F}_t$, a property which can be characterized as follows:

$$B \in \mathcal{F}_t \wedge B \subseteq A \quad \Rightarrow \quad B = \emptyset \vee B = A$$

An atom $A_i \in \mathcal{F}_t$ splits into a collection of mutually disjoint atoms B_j, $j = 1, 2, \ldots, \nu(t, A_i)$ in $\mathcal{F}_{t+1}$[20]*. If $A_i \in \mathcal{F}_t$ is an atom then $S_t(\omega)$ is constant across all $\omega \in A_i$. Hence, the elements of $\mathcal{M}_t$ are vectors with groupwise identical entries, $X(B_1)$, $X(B_2)$,..., $X(B_{\nu(t,A_i)})$, and the vector S_t is characterized as a vector with groupwise identical entries $S_t(A_i)$. The value of the bank account at time $t+1$ is similarly a vector with groupwise identical entries $B_{t+1}(A_i)$ because of the predictability requirement; and this also goes for the bank account after tax $B^{a.t.}_{t+1}$.*

Given these observations the subspace $\mathcal{M}_t$ is obviously the the same subspace as the one spanned by the vectors $(1-T_t)S_{t+1} + T_t S_t$ and S_t for any given $\omega \in A_i$. This is so because conditioned on each atom the elements of the vector S_t is proportional to the value of the riskless asset. Hence, the set of attainable claims is invariant to a linear and symmetric taxation schedule. ∎

Proof (Theorem 2) *The important parts of theorem 2 are statements 1. and 2. and the conditions in (40). These provide necessary and sufficient conditions on the parameters of the taxation system in order to ensure valuation neutrality.*

The statements in 3. and the conditions (43)-(45) are merely rewritings of – and, hence, equivalent to – the conditions in (40). In the same way the statements in 4. are restatements of the conditions in (40) once it is observed that the tax account is reset to value zero at any intermediate realization. After such an intermediate realization the clock as well as the discounting are both restarted.

The statement in 5. is a trivial consequence of the conditions in 1. and the equivalent conditions in 3. Since a given linear and symmetric neutral taxation schedule is uniquely characterized by the sequence $(K^0_{n,0}, K^0_{n,1}, \ldots, K^0_{n,n})$ it follows from (40) that any convex combination of two such schedules is again a linear and symmetric neutral taxation schedule[21]*. It can also be deduced directly from the basic requirement in (41).*

The statement in 6. is directly related to the statement in 2., which can be interpreted as an "invertibility property", and the proof is similar. To the extent that the risk exposure after tax is not nullified completely, it is possible to switch between the after-tax and the before-tax relations.

[20] The function ν is called the splitting function and the value $\nu(t, A)$ is called the splitting index. See Jensen and Nielsen (1996) for further details.

[21] As a matter of fact in pure mathematical terms an affine combination is sufficient. However, this might result in taxation schedules that are economically meaningless.

As for 1. consider the representation (32) which is equivalent to (41). Then the first half of the conditions in (40) is obvious – the first sum in (32) vanishes except for the term $j=t$. It remains to show that the rest vanishes due to the second part of the conditions in (40). This follows from the following sequence of calculations:

$$\frac{\theta^t S_t}{B^t} = E^Q\left[\frac{\theta^n S_n}{B^n}\,\middle|\,\mathcal{F}_t\right] \quad\wedge\quad \frac{K^0_{t,t}}{B^{a.t.}_t} = \sum_{j=t}^{n}\frac{K^0_{n,j}}{B^{a.t.}_n}\frac{B_j}{B_t} \quad\Rightarrow \tag{A.5}$$

$$\frac{K^0_{t,t}}{B^{a.t.}_t}\theta^t S_t = E^Q\left[\sum_{j=t}^{n}\frac{K^0_{n,j}}{B^{a.t.}_n}B_j\frac{\theta^n S_n}{B^n}\,\middle|\,\mathcal{F}_t\right] \quad\Rightarrow \tag{A.6}$$

$$\frac{K^0_{t,t}}{B^{a.t.}_t}\theta^t S_t = \sum_{j=t}^{n}E^Q\left[E^Q\left[\frac{K^0_{n,j}}{B^{a.t.}_n}B_j\frac{\theta^n S_n}{B^n}\,\middle|\,\mathcal{F}_j\right]\,\middle|\,\mathcal{F}_t\right] \tag{A.7}$$

$$= \sum_{j=t}^{n}E^Q\left[\frac{K^0_{n,j}}{B^{a.t.}_n}B_jE^Q\left[\frac{\theta^n S_n}{B_n}\,\middle|\,\mathcal{F}_j\right]\,\middle|\,\mathcal{F}_t\right] \tag{A.8}$$

$$= \sum_{j=t}^{n}E^Q\left[\frac{K^0_{n,j}}{B^{a.t.}_n}B_j\frac{\theta^j S_j}{B_j}\,\middle|\,\mathcal{F}_t\right] \tag{A.9}$$

$$= \sum_{j=t}^{n}E^Q\left[\frac{K^0_{n,j}}{B^{a.t.}_n}\theta^j S_j\,\middle|\,\mathcal{F}_t\right] \tag{A.10}$$

This ends the proof of 1.

In order to prove 2. an induction argument is applied. Consider first an investment strategy such that $\theta^1 S_0 \neq 0$. For $m=1$, (41) takes on the form

$$\theta^1 S_0 = \frac{K^0_{10}}{B^{a.t.}_1}\theta^1 S_0 + \frac{K^0_{11}}{B^{a.t.}_1}E^Q[\theta^1 S_1] \tag{A.11}$$

Inserting the special investment strategy $\theta^1 S_1 = \theta^1 S_0 B_1$, observing that $B_1 \in \mathcal{F}_0$, leads to (40) for this particular case:

$$\theta^1 S_0 = \frac{K^0_{10}}{B^{a.t.}_1}\theta^1 S_0 + \frac{K^0_{11}}{B^{a.t.}_1}\theta^1 S_0 B_1 \quad\Leftrightarrow\quad 1 = \frac{K^0_{10}}{B^{a.t.}_1} + \frac{K^0_{11}}{B^{a.t.}_1}B_1 \tag{A.12}$$

Given this relation, we can substitute into (A.11) and get

$$0 = \frac{K^0_{11}}{B^{a.t.}_1}B_1\left(\theta^1 S_0 - E^Q\left[\frac{\theta^1 S_1}{B_1}\right]\right) \tag{A.13}$$

Whenever $K^0_{11} \neq 0$ the pricing relation before tax must be fulfilled with the equivalent martingale measure Q. This establishes the beginning of an induction proof with $t=0$ and $m=1$.

Consider next the case $m=2$. For any self-financing portfolio strategy we must have the pricing relation for $t=1$:

$$\frac{K^0_{10}}{B^{a.t.}_1}\theta^1 S_0 + \frac{K^0_{11}}{B^{a.t.}_1}\theta^1 S_1 = E^Q\left[\frac{K^0_{20}}{B^{a.t.}_2}\theta^1 S_0 + \frac{K^0_{21}}{B^{a.t.}_2}\theta^1 S_1 + \frac{K^0_{22}}{B^{a.t.}_2}\theta^2 S_2\middle|\mathcal{F}_1\right]$$

$$= \frac{K^0_{20}}{B^{a.t.}_2}\theta^1 S_0 + \frac{K^0_{21}}{B^{a.t.}_2}\theta^1 S_1 + \frac{K^0_{22}}{B^{a.t.}_2}E^Q\left[\theta^2 S_2 \mid \mathcal{F}_1\right] \quad\Leftrightarrow \tag{A.14}$$

$$\left(\frac{K^0_{10}}{B^{a.t.}_1} - \frac{K^0_{20}}{B^{a.t.}_2}\right)\theta^1 S_0 = \left(\frac{K^0_{21}}{B^{a.t.}_2} - \frac{K^0_{11}}{B^{a.t.}_1}\right)\theta^1 S_1 + \frac{K^0_{22}}{B^{a.t.}_2}E^Q\left[\theta^2 S_2 \mid \mathcal{F}_1\right] \tag{A.15}$$

The following is a special case of (A.15), corresponding to an investment strategy with $\theta^2 S_2 = \theta^1 S_1 (B_2/B_1)$:

$$\left(\frac{K^0_{10}}{B^{a.t.}_1} - \frac{K^0_{20}}{B^{a.t.}_2}\right)\theta^1 S_0 = \left(\frac{K^0_{21}}{B^{a.t.}_2} + \frac{K^0_{22}}{B^{a.t.}_2}\frac{B_2}{B_1} - \frac{K^0_{11}}{B^{a.t.}_1}\right)\theta^1 S_1 \tag{A.16}$$

We also have the pricing relation for $t=0$:

$$\theta^1 S_0 = E^Q\left[\frac{K^0_{20}}{B^{a.t.}_2}\theta^1 S_0 + \frac{K^0_{21}}{B^{a.t.}_2}\theta^1 S_1 + \frac{K^0_{22}}{B^{a.t.}_2}\theta^2 S_2\right] \tag{A.17}$$

Assume that a self-financing investment strategy can be constructed such that $\theta^1 S_0 = 0$, but $\theta^1 S_1 \neq 0 \ \forall \omega \in \Omega$. This is verified below. Then the following identity between random variables immediately follows from (A.16):

$$\frac{K^0_{11}}{B^{a.t.}_1} = \frac{K^0_{21}}{B^{a.t.}_2} + \frac{K^0_{22}}{B^{a.t.}_2}\frac{B_2}{B_1} \tag{A.18}$$

Choosing any alternative investment strategy with $\theta^1 S_0 \neq 0$ as well as $\theta^1 S_1 \neq 0$ leads to

$$\frac{K^0_{20}}{B^{a.t.}_2} = \frac{K^0_{10}}{B^{a.t.}_1} \tag{A.19}$$

This proves the necessity of the relations (40) for $m=2$ and $t=0,1$. Additionally, we can rewrite (A.14) after cancellation as

$$0 = \frac{K^0_{22}}{B^{a.t.}_2}B_2\left[E^Q\left[\frac{\theta^2 S_2}{B_2}\middle|\mathcal{F}_1\right] - \frac{\theta^1 S_1}{B_1}\right] \tag{A.20}$$

When $K^0_{22} \neq 0$ we have the equivalent before tax relation

$$\frac{\theta^1 S_1}{B_1} = E^Q\left[\frac{\theta^2 S_2}{B_2}\middle|\mathcal{F}_1\right] \tag{A.21}$$

The regularity condition – that a non-trivial investment strategy exists – means that at any atom $A \in \mathcal{F}_t$ it is possible to find a portfolio such that the realized return at any atom $B \in \mathcal{F}_{t+1}$, $B \subseteq A$, is never equal to the riskless rate of return. This also implies that is always possible to find a portfolio with zero net investment,[22] such that the realized value after one period is never zero in any state. The recipe is to combine a non-trivial investment strategy at time t with a short position of identical value in the bank account. When $t = 0$ the result of such an investment strategy is never zero in any state at time 1. Or, stated alternatively, an investment strategy always exists such that $\theta^1 S_0 = 0$, but $\theta^1 S_1 \neq 0 \ \forall \omega \in \Omega$.[23] Similar reasoning can be applied at any point in time t by simply investing in nothing until time t.

Assume now that the necessity is established for (m,t), $m=0,1,\ldots,\bar{m}$, $t=0,1,\ldots,m$. Consider the relation (40) for neighboring indices $\bar{m}$ and $\bar{m}+1$:

$$\frac{1}{B^{a.t.}_{\bar{m}}}\sum_{j=0}^{\bar{m}} K^0_{\bar{m},j}\theta^j S_j = E^Q\left[\frac{1}{B^{a.t.}_{\bar{m}+1}}\sum_{j=0}^{\bar{m}+1} K^0_{\bar{m}+1,j}\theta^j S_j\middle|\mathcal{F}_{\bar{m}}\right] =$$

[22] A *zenip*, cf. Harrison and Kreps (1979).

[23] This is one variant of the insight found in Ross (1976).

$$\frac{1}{B^{a.t.}_{\bar{m}+1}}\left(\sum_{j=0}^{\bar{m}} K^0_{\bar{m}+1,j}\theta^j S_j + K^0_{\bar{m}+1,\bar{m}+1} E^Q\left[\theta^{\bar{m}+1} S_{\bar{m}+1}\middle|\mathcal{F}_{\bar{m}}\right]\right) \tag{A.22}$$

Following the same procedure as above we can find a self-financing portfolio strategy such that

$$\theta^1 S_0 = \theta^1 S_1 = \ldots = \theta^{\bar{m}-1} S_{\bar{m}-1} = 0, \quad \theta^{\bar{m}} S_{\bar{m}} \neq 0, \theta^{\bar{m}+1} S_{\bar{m}+1} = \theta^{\bar{m}} S_{\bar{m}} \frac{B_{\bar{m}+1}}{B_{\bar{m}}} \tag{A.23}$$

This establishes the necessity of the relation

$$\frac{K^0_{\bar{m},\bar{m}}}{B^{a.t.}_{\bar{m}}} = \frac{K^0_{\bar{m}+1,\bar{m}}}{B^{a.t.}_{\bar{m}+1}} + \frac{K^0_{\bar{m}+1,\bar{m}+1}}{B^{a.t.}_{\bar{m}+1}} \frac{B_{\bar{m}+1}}{B_{\bar{m}}} \tag{A.24}$$

By backwards substitution one can generate all the desired relations in the second half of (40). Additionally, when $K^0_{\bar{m}+1,\bar{m}+1} \neq 0$ the pre-tax relation, cf. (A.21) follows along the lines of reasoning in (A.20):

$$\frac{\theta^{\bar{m}} S_{\bar{m}}}{B_{\bar{m}}} = E^Q\left[\frac{\theta^{\bar{m}+1} S_{\bar{m}+1}}{B_{\bar{m}+1}}\middle|\mathcal{F}_{\bar{m}}\right] \tag{A.25}$$

Working backwards step by step and in turn construct investment strategies in accordance with the recipe in (A.23) leads to the necessity of the relations

$$\frac{K^0_{\bar{m},j}}{B^{a.t.}_{\bar{m}}} = \frac{K^0_{\bar{m}+1,j}}{B^{a.t.}_{\bar{m}+1}} \tag{A.26}$$

for $j = \bar{m}-1,\ \bar{m}-2,\ \ldots,\ 1, 0$. This ends the induction proof of 2.

Finally, we prove statement 6. The set of attainable claims after tax at time m is the subspace $\sum_{j=0}^m K^0_{m,j}\theta^j S_j$ spanned by all self-financing portfolios $\theta^j,\ j=0,1,\ldots,m$. By using the relations in (40) this subspace can be rewritten as

$$\begin{aligned}\sum_{j=0}^{m} K^0_{m,j}\theta^j S_j &= \sum_{j=0}^{m-1} K^0_{m,j}\theta^j S_j + K^0_{m,m}\theta^m S_m \\ &= \left(\frac{B^{a.t}_m}{B^{a.t.}_{m-1}}\right)\sum_{j=0}^{m-1} K^0_{m-1,j}\theta^j S_j + K^0_{m,m}\theta^m S_m \end{aligned} \tag{A.27}$$

When $K^0_{m,m} \neq 0$ the argument is identical to the analogous argument in the proof of theorem 1. We use an induction argument and assume that it has been established that the set of attainable claims at time $m-1$ is invariant to the linear taxation system $K^0_{m-1,j},\ j=0,1,\ldots,m-1$. The initialization of this induction argument is straightforward for $m=1$.

Analogous to the argument in the proof of theorem 1 above the elements of the vectors in this subspace have groupwise identical entries on each atom of $\mathcal{F}_t$. Hence, any desired element of the form $\psi^m S_m$ can be found by

1. *setting $\theta^m = \psi^m / K^0_{m,m}$*
2. *work backwards to find the self-financing portfolio policy replacating this*
3. *adjust the position in the bank account to correct for the first term.*

∎

B The Auerbach&Bradford analysis

The Bradford (1995) analysis imposes a linear cash flow taxation system as follows:

$$A_n^\theta = \left[1-(1-g)\frac{B_n^{a.t.}}{B_D^{a.t.}}\frac{B_D}{B_n}\right]\theta^n S_n - B_n^{a.t.}\left[1-(1-g)\frac{B_D}{B_D^{a.t.}}\right]\theta^1 S_0$$

where $0 \le D \le n$ is the socalled "gains recognition date" and g is the "gains taxation rate". The after tax payment at maturity − provided no wash sale takes place − is

$$\theta^n S_n(1-g)\frac{B_n^{a.t.}}{B_D^{a.t.}}\frac{B_D}{B_n} + B_n^{a.t.}\left[1-(1-g)\frac{B_D}{B_D^{a.t.}}\right]\theta^1 S_0$$

In our notation $H = \left[1-(1-g)\frac{B_D}{B_D^{a.t.}}\right]$ and $H = (1-g)\frac{B_D^{a.t.}}{B_D}$.

This scheme is not able to handle situations with stochastic interest rates or an otherwise stochastic, although predictable, development in the bank account. H must be known a priori in order for this scheme to work and it cannot be extended to encompass this type of uncertainty.

The same is true for the original Brown tax. ∎

C Capital injections

In this appendix some details from section 6 are written out.

Model	Investment	Credit	After tax value at maturity
Auerbach(1991)	1	$1-\frac{B_t^{a.t.}}{B_t}$	$\underbrace{\frac{B_n}{B_t}\frac{B_n^{a.t.}}{B_n}}_{\text{(unit investment)}} + \underbrace{\frac{B_n^{a.t.}}{B_t^{a.t.}}\left(1-\frac{B_t^{a.t.}}{B_t}\right)}_{\text{(tax credit)}} = \frac{B_n^{a.t.}}{B_t^{a.t.}}$
Mark-to-market	1	T_{t-1}	$(1-T_{n-1})\frac{B_n}{B_t} + \sum_{j=t}^{n-1}\left[\frac{T_j}{B_{j+1}^{a.t.}} - \frac{T_{j-1}}{B_j^{a.t.}}\right]\frac{B_j B_n^{a.t.}}{B_t} + T_{t-1}\frac{B_n^{a.t.}}{B_t^{a.t}} = (1-T_{n-1})\frac{B_n}{B_t} + T_{t-1}\frac{B_n^{a.t.}}{B_t^{a.t}} + \sum_{j=t}^{n-1}\left[\frac{T_j}{(B_{j+1}^{a.t.}/B_t^{a.t.})} - \frac{T_{j-1}}{(B_j^{a.t.}/B_t^{a.t.})}\right]\frac{B_j}{B_t}\frac{B_n^{a.t.}}{B_t^{a.t.}}$

Table 2: After tax value at maturity of a capital inflow

First we verify − for two key models − that the proposed tax crediting mechanism for incremental capital injections has the same effect in terms of valuation as resetting the clock for the holding period. The capital inflow at time t is assumed invested in the bank account and the initial tax credit is carried forward via the bank account after tax. Table 2 shows that the value after tax of a capital injection at time t is the same as the value that would occur if the investment is taxed in accordance with the stipulated taxation principle and with its own clock.

The verification for a more general investment policy is mostly a matter of using more complicated notation.

Now we prove that the latter part of (74) has t-value zero:

$$
E^Q\left[\frac{B_n^{a.t.}}{B_t^{a.t.}}\left(\sum_{j=0}^{t}K_{t,j}^0\theta^j S_j\right)-\sum_{j=0}^{n}K_{n,j}^0\theta^j S_j\middle|\mathcal{F}_t\right]=
$$

$$
E^Q\left[\left(\sum_{j=0}^{t}\left(\frac{B_n^{a.t.}}{B_t^{a.t.}}K_{t,j}^0\right)\theta^j S_j\right)-\sum_{j=0}^{n}K_{n,j}^0\theta^j S_j\middle|\mathcal{F}_t\right]=
$$

$$
E^Q\left[\left(\frac{B_n^{a.t.}}{B_t^{a.t.}}K_{t,t}^0\right)\theta^t S_t-\sum_{j=t}^{n}K_{n,j}^0\theta^j S_j\middle|\mathcal{F}_t\right]=E^Q\left[\sum_{j=t}^{n}K_{n,j}^0\frac{B_j}{B_t}\theta^t S_t-\sum_{j=t}^{n}K_{n,j}^0\theta^j S_j\middle|\mathcal{F}_t\right]=
$$

$$
E^Q\left[\sum_{j=t}^{n}K_{n,j}^0\left(\frac{B_j}{B_t}\theta^t S_t-\theta^j S_j\right)\middle|\mathcal{F}_t\right]=E^Q\left[\sum_{j=t}^{n}E^Q\left[K_{n,j}^0\left(\frac{B_j}{B_t}\theta^t S_t-\theta^j S_j\right)\middle|\mathcal{F}_j\right]\middle|\mathcal{F}_t\right]=
$$

$$
E^Q\left[\sum_{j=t}^{n}K_{n,j}^0 B_j E^Q\left[\frac{\theta^t S_t}{B_t}-\frac{\theta^j S_j}{B_j}\middle|\mathcal{F}_j\right]\middle|\mathcal{F}_t\right]=E^Q\left[\sum_{j=t}^{n}K_{n,j}^0 B_j\left[\frac{\theta^t S_t}{B_t}-E^Q\left(\frac{\theta^j S_j}{B_j}\middle|\mathcal{F}_j\right)\right]\middle|\mathcal{F}_t\right] \tag{C.28}
$$

Since each individual term is zero by the martingale property of discounted prices it is proven that the latter part of (74) has t-value zero.